

Log-like? Identified ATEs defined with zero-valued outcomes are (arbitrarily) scale-dependent*

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Abstract

Economists frequently estimate average treatment effects (ATEs) for transformations of the outcome that are well-defined at zero but behave like $\log(y)$ when y is large (e.g., $\log(1 + y)$, $\operatorname{arcsinh}(y)$). We show that these ATEs depend arbitrarily on the units of the outcome, and thus cannot be interpreted as percentage effects. Moreover, we prove that when the outcome can equal zero, there is no parameter of the form $E_P[g(Y(1), Y(0))]$ that is point-identified and unit-invariant. We discuss sensible alternative target parameters for settings with zero-valued outcomes that relax at least one of these requirements.

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1 Introduction

When the outcome of interest Y is strictly positive, researchers often estimate the average treatment effect (ATE) in logs, $E[\log(Y(1)) - \log(Y(0))]$, which has the appealing feature that its units approximate percentage changes in the outcome.¹ A practical challenge in many economic settings, however, is that the variable of interest may sometimes equal zero, and thus the ATE in logs is not well-defined. When this is the case, it is common for researchers to estimate ATEs for alternative transformations of the outcome such as $\log(1+Y)$ or $\operatorname{arcsinh}(Y) = \log(\sqrt{1+Y^2} + Y)$, which behave similarly to $\log(Y)$ for large values of Y but are well-defined at zero. The treatment effects for these alternative transformations are typically interpreted like the ATE in logs, i.e. as (approximate) percentage effects. For example, we found that 10 of the 11 papers published in the *American Economic Review* since 2018 that estimate a treatment effect for $\operatorname{arcsinh}(Y)$ interpret the result as a percentage effect or elasticity.²

The main point of this paper is that identified ATEs that are well-defined with zero-valued outcomes cannot be interpreted as percentage effects, at least if one imposes the logical requirement that a percentage effect does not depend on the baseline units in which the outcome is measured (e.g. dollars, cents, or yuan).

Our first main result shows that if $m(y)$ is a function that behaves like $\log(y)$ for large values of y but is defined at zero, then the ATE for $m(Y)$ will be *arbitrarily sensitive* to the units of Y , in the sense that one can obtain an ATE of *any magnitude* by re-scaling the units. Specifically, we consider continuous, increasing functions $m : [0, \infty) \rightarrow \mathbb{R}$ that approximate $\log(y)$ for large values of y , in the sense that

$$\lim_{y \rightarrow \infty} \frac{m(y)}{\log(y)} = 1. \quad (1)$$

The common $\log(1+y)$ and $\operatorname{arcsinh}(y)$ transformations satisfy this property. We show that if treatment affects the extensive margin (i.e. $P(Y(1) = 0) \neq P(Y(0) = 0)$), then one can obtain any magnitude for the ATE for $m(Y)$ by rescaling the outcome by some positive factor a . It is therefore inappropriate to interpret the ATE for $m(Y)$ as a percentage, since a percentage is inherently a unit-invariant quantity.

The intuition for this result is that a “proportional” treatment effect is not well-defined for individuals whose outcome moves from zero to non-zero values. Any average treatment effect that is well-defined with zero-valued outcomes must therefore (implicitly) assign a value for a change along the extensive margin relative to proportional changes along the intensive margin. For logarithm-like transformations $m(\cdot)$, the relative importance of the extensive margin is determined implicitly by the units of Y . If we re-scale the units so that the non-zero values of Y are typically very large, then changing an individual’s outcome from zero to a typical non-zero value of the outcome has a very large impact on $m(Y)$, and so the ATE places an extremely large weight on the extensive margin.

¹In particular, $\log(Y(1)/Y(0)) \approx \frac{Y(1)-Y(0)}{Y(0)}$ when $Y(1)/Y(0) \approx 1$.

²We found 17 papers overall using $\operatorname{arcsinh}(Y)$ as an outcome variable, of which 11 interpret the units; see [Appendix Table 1](#).

When treatment has an extensive margin effect, the ATE can thus be made large in magnitude by making the units of Y large. By contrast, if we re-scale the units of Y such that all of the non-zero values of Y are close to zero, then $m(Y) \approx m(0)$, and thus the ATE for $m(Y)$ will be small. By varying the units of the outcome, we can thus obtain any arbitrary magnitude for the ATE.

We illustrate the practical importance of these results by systematically replicating recent papers published in the *American Economic Review* that estimate treatment effects for the arcsinh transformation of the outcome. In line with our theoretical results, we find that treatment effect estimates using $\text{arcsinh}(Y)$ are sensitive to the scaling of the outcome when there is an effect on the extensive margin. In half of the papers that we replicated, multiplying the original outcome by a factor of 100 (e.g. converting from dollars to cents) changes the estimated treatment effect by more than 100% of the original estimate.

What, then, are alternative options in settings with zero-valued outcomes? Our second main result delineates the possibilities. We show that when there are zero-valued outcomes, there is no treatment effect parameter that satisfies all three of the following properties:

- (a) The parameter is an average of a function of the potential outcomes, $\theta_g = E_P[g(Y(1), Y(0))]$, where the function g is increasing in $Y(1)$.
- (b) The parameter is invariant to re-scaling of the units of the outcome (i.e. $g(y_1, y_0) = g(ay_1, ay_0)$).
- (c) The parameter is point-identified from the marginal distributions of the potential outcomes.

This “trilemma” implies that any approach that accommodates zero-valued outcomes must necessarily jettison (at least) one of the three properties above. Which one the researcher prefers to forgo will generally depend on their motivation for using a log-like transformation in the first place.

To that end, we conclude by highlighting a menu of approaches that may be attractive depending on the researcher’s core motivation. First, suppose the researcher is interested in obtaining a causal parameter with an intuitive “percentage” interpretation. Then it may be natural to consider a normalized parameter outside of the class $E_P[g(Y(1), Y(0))]$. One option is $\theta_{\text{ATE}\%} = \frac{E[Y(1) - Y(0)]}{E[Y(0)]}$, the ATE in levels as a percentage of the baseline mean. A second option is the ATE for a normalized outcome of the form $\tilde{Y} = Y/X$, e.g. the employment-to-population ratio (if Y is employment and X is population). Next, suppose the researcher would like to capture concave preferences over the outcome; for example, the researcher might consider income gains to be more meaningful for individuals who are initially poor. In this case, it is natural to directly specify how much the researcher values a change along the extensive margin relative to the intensive margin—e.g., that a change from 0 to 1 is worth an x percent change along the intensive margin. Finally, suppose the researcher is interested in separately understanding the effects of the treatment along both the intensive and extensive margins. In this case, the researcher may target separate parameters for the two margins—e.g., $E[\log(Y(1)) - \log(Y(0)) \mid Y(1) > 0, Y(0) > 0]$, the average effect for individuals with positive outcomes under both treatments, captures the intensive margin. Separate effects for the two margins are not generally point-identified, but can be bounded using the method in Lee (2009) or point-identified with additional assumptions (Zhang, Rubin and Mealli, 2009).

Related work. Previous work has illustrated in simulations or selected empirical applications that results for particular transformations such as $\log(1 + Y)$ or $\operatorname{arcsinh}(Y)$ may be sensitive to the units of the outcome (Aihounton and Henningsen, 2021; de Brauw and Herskowitz, 2021). We complement this work by proving that scale-dependence is a necessary feature of *any* identified ATE that is well-defined with zero-valued outcomes, and that the dependence on units is arbitrarily bad for transformations that approximate $\log(Y)$ for large values of Y . We also complement previous empirical examples by providing a systematic analysis of the sensitivity to scaling for papers in the *American Economic Review* using $\operatorname{arcsinh}(Y)$.

In concurrent work, Mullahy and Norton (2022) show that the marginal effects from linear regressions using $\log(1 + Y)$ or $\operatorname{arcsinh}(Y)$ are sensitive to the scaling of the outcome. Specifically, they show that the marginal effects converge to those of either a levels regression or a (normalized) linear probability model, depending on whether the units are made small or large. Since the marginal effects of linear models correspond with ATEs under unconfoundedness, these results help provide intuition for our result that the ATE for any log-like transformation is *arbitrarily* sensitive to the scaling of the outcome. We show that sensitivity to scaling is an issue beyond the class of linear models using the $\log(1 + Y)$ or $\operatorname{arcsinh}(Y)$ transformations considered in Mullahy and Norton (2022), however. Our results thus imply that the scale-dependence cannot be avoided by finding a better transformation or a different estimator.

Other previous work has considered the interpretation of regressions using $\operatorname{arcsinh}(Y)$ or $\log(1 + Y)$ from the perspective of structural equations models, as opposed to the potential outcomes model considered here. This literature has reached diverging conclusions: For example, Bellemare and Wichman (2020) conclude that coefficients from $\operatorname{arcsinh}(Y)$ regressions have an interpretation as a semi-elasticity, while Cohn, Liu and Wardlaw (2022) conclude that these estimators are inconsistent and advocate for Poisson regression instead. In Appendix D, we show that these diverging conclusions stem from the fact that the structural equations considered in these papers implicitly impose different restrictions on the potential outcomes—some of which are incompatible with zero-valued outcomes—and consider different target causal parameters. This highlights the value of a potential outcomes framework such as ours, which makes transparent what causal parameters are identifiable and what properties they can have.

1.1 Setup and notation

Let $D \in \{0, 1\}$ be a binary treatment and let $Y \in [0, \infty)$ be a weakly positively-valued outcome.³ We assume that $Y = DY(1) + (1 - D)Y(0)$, where $Y(1)$ and $Y(0)$ are respectively the potential outcomes under treatment and control. We suppose that in some population of interest, $(Y(1), Y(0)) \sim P$ for some (unknown) joint distribution P . We denote the marginal distribution of $Y(d)$ under P by $P_{Y(d)}$ for $d = 0, 1$. We assume that neither $P_{Y(0)}$ nor $P_{Y(1)}$ is a degenerate distribution at zero.

³See Appendix C.2 for extensions of our results to settings with continuous treatments.

2 Sensitivity to scaling for transformations that behave like $\log(Y)$

We first consider average treatment effects of the form $\theta = E_P[m(Y(1)) - m(Y(0))]$ for an increasing function m . We are interested in how θ changes if we change the units of Y by a factor of a . That is, how does

$$\theta(a) = E_P[m(aY(1)) - m(aY(0))]$$

depend on a ? Setting $a = 100$, for example, might correspond with a change in units between dollars and cents.

We consider functions $m(y)$ that behave like $\log(y)$ for large values of y , in the sense of (1). This property is satisfied by $\log(1 + y)$ and $\operatorname{arcsinh}(y)$, for example. Our first main result then shows that if the treatment affects the extensive margin, then $|\theta(a)|$ can be made to take any desired value through the appropriate choice of a .

Proposition 1. *Suppose that:*

1. (The function m is continuous and increasing) $m : [0, \infty) \rightarrow \mathbb{R}$ is a continuous, weakly increasing function.
2. (The function m behaves like \log for large values) $m(y)/\log(y) \rightarrow 1$ as $y \rightarrow \infty$.
3. (Treatment affects the extensive margin) $P(Y(1) = 0) \neq P(Y(0) = 0)$.
4. (Finite expectations) $E_{P_{Y(d)}}[|\log(Y(d))| \mid Y(d) > 0] < \infty$ for $d = 0, 1$.⁴

Then, for every $\theta^* \in (0, \infty)$, there exists an $a > 0$ such that $|\theta(a)| = \theta^*$. In particular, $\theta(a)$ is continuous with $\theta(a) \rightarrow 0$ as $a \rightarrow 0$ and $|\theta(a)| \rightarrow \infty$ as $a \rightarrow \infty$.

Proposition 1 casts serious doubt on the interpretation of ATEs for functions like $\log(1 + Y)$ or $\operatorname{arcsinh}(Y)$ as (approximate) percentage effects. While a percent (or log point) is entirely invariant to scaling, Proposition 1 shows that, in sharp contrast, the ATEs for these transformations are arbitrarily dependent on units.

Remark 1 (ATEs for $\log(c + Y)$). In some settings, researchers consider the ATE for $\log(c + Y)$ and investigate sensitivity to the parameter c . Observe that $\log(1 + aY) = \log(a(1/a + Y)) = \log(a) + \log(1/a + Y)$, and thus the ATE for $\log(1 + aY)$ is equal to the ATE for $\log(1/a + Y)$. Hence, varying the constant term for $\log(c + Y)$ is equivalent to varying the scaling of the outcome when using $m(y) = \log(1 + y)$. Proposition 1 thus implies that if treatment affects the extensive margin, one can obtain any desired magnitude for the ATE for $\log(c + Y)$ via the choice of c .

2.1 Intuition for Proposition 1

The result in Proposition 1 intuitively derives from the fact that a “percentage” treatment effect is not well-defined for individuals who have $Y(0) = 0$ but $Y(1) > 0$, or vice versa. Any ATE that

⁴This assumption simply ensures that $E_{P_{Y(d)}}[|m(aY(d))| \mid Y > 0]$ exists for all values of $a > 0$.

is well-defined with zero-valued outcomes must implicitly determine how much weight to place on changes along the extensive margin relative to proportional changes along the intensive margin.

When $m(Y)$ behaves like $\log(Y)$ for large values of Y , the importance of the extensive margin is implicitly determined by the units of Y . For intuition, suppose that we re-scale the outcomes so that the non-zero values of Y are very large. Then for an individual for whom treatment changes the outcome from zero to non-zero, the treatment effect will be very large, since $m(Y(1)) \gg m(Y(0)) = m(0)$. Extensive margin treatment effects thus have a large impact on the ATE when the units of Y are made large. By contrast, changing the units of Y does not change the importance of treatment effects along the intensive margin by much, since for $Y(1), Y(0) > 0$, we have that $m(Y(1)) - m(Y(0)) \approx \log(Y(1)/Y(0))$, which does not depend on the units of the outcome.

To see the roles of the extensive and intensive margins more formally, for simplicity consider the case where $P(Y(1) = 0, Y(0) > 0) = 0$, so that, for example, everyone who has positive income without receiving a training also has positive income when receiving the training.⁵ Then, by the law of iterated expectations, we can write

$$E[m(aY(1)) - m(aY(0))] = P(Y(1) > 0, Y(0) > 0) \underbrace{E_P[m(aY(1)) - m(aY(0)) \mid Y(1) > 0, Y(0) > 0]}_{\text{Intensive margin}} \\ + P(Y(1) > 0, Y(0) = 0) \underbrace{E_P[m(aY(1)) - m(0) \mid Y(1) > 0, Y(0) = 0]}_{\text{Extensive margin}}.$$

When a is large, $m(ay) \approx \log(ay)$ for non-zero values of y , and thus the intensive margin effect in the previous display is approximately equal to $E_P[\log(Y(1)) - \log(Y(0)) \mid Y(1) > 0, Y(0) > 0]$, the treatment effect in logs for individuals with positive outcomes under both treatment and control. This, of course, does not depend on the scaling of the outcome. However, the extensive margin effect grows with a , since $m(aY(1)) \approx \log(a) + \log(Y(1))$ is increasing in a while $m(0)$ does not change. Thus, as a grows large, the ATE for $m(aY)$ places more and more weight on the extensive margin effect of the treatment relative to the intensive margin. We can therefore make $|\theta(a)|$ arbitrarily large by sending $a \rightarrow \infty$. By contrast, if $a \approx 0$, then $m(aY(d)) \approx 0$ with very high probability, and thus the ATE for $m(aY)$ is approximately equal to 0.

2.2 Additional remarks and extensions

Remark 2 (Finite changes in scaling). [Proposition 1](#) shows that any magnitude of $|\theta(a)|$ can be achieved via the appropriate choice of a . How much does $\theta(a)$ change for finite changes in the scaling a ? [Proposition 4](#) in the appendix shows that the change in the ATE from multiplying the outcome by a large factor a is approximately $\log(a)$ times the treatment effect on the extensive margin,⁶

$$E_P[m(a \cdot Y(1)) - m(a \cdot Y(0))] = (P(Y(1) > 0) - P(Y(0) > 0)) \cdot \log(a) + o(\log(a)). \quad (2)$$

⁵A related argument goes through without this restriction, but now there are two extensive margins, one for individuals with $Y(1) > 0 = Y(0)$, and the other for those with $Y(0) > Y(1) = 0$.

⁶We say $f(a) = o(g(a))$ if $\lim_{a \rightarrow \infty} |f(a)/g(a)| = 0$. That is, as $a \rightarrow \infty$, $|f(a)|$ grows strictly slower than $|g(a)|$.

Thus, the ATE for $m(Y)$ will tend to be more sensitive to finite changes in scale the larger is the extensive margin treatment effect.

Remark 3 (Extension to continuous treatments). We focus on ATEs for binary treatments for expositional simplicity, although similar results apply with continuous treatments. In [Appendix C.2](#), we show that when d is a continuous treatment, any treatment effect contrast that averages $m(aY(d))$ across possible values of d (i.e. a parameter of the form $\int \omega(d)E[m(aY(d))]$) is sensitive to scaling when there is an extensive margin effect.

Remark 4 (Extension to OLS estimands). It is worth noting that the results in this section show that population ATEs for $m(Y)$ are sensitive to the units of Y . These results are about *estimands*, and thus any consistent *estimator* of the ATE for $m(Y)$ will be sensitive to scaling (at least asymptotically). Thus, our results apply to ordinary least squares (OLS) estimators when they have a causal interpretation, but also to non-linear estimators such as inverse-probability weighting or doubly-robust methods. Nevertheless, given the prominence of OLS in applied work, and the fact that OLS is sometimes used for non-causal estimands, in [Appendix C.3](#) we provide a result specifically on the scale-sensitivity of the population regression coefficient for a random variable of the form $m(Y)$ on an arbitrary random variable X . Our results shows that the coefficients on X will be arbitrarily sensitive to the scaling of Y when the coefficients of a regression of $\mathbb{1}[Y > 0]$ on X are non-zero. Thus, the OLS estimand using a logarithm-like function on the left-hand side will be sensitive to scaling even when it does not have a causal interpretation.

Remark 5 (When most values are large). Researchers often have the intuition that if most of the values of the outcome are large, then ATEs for transformations like $\log(1 + Y)$ or $\operatorname{arcsinh}(Y)$ will approximate elasticities, since $m(Y) \approx \log(Y)$ for most values of Y . Indeed, in an influential paper, [Bellemare and Wichman \(2020\)](#) recommend that researchers using the $\operatorname{arcsinh}(Y)$ transformation should transform the units of their outcome so that most of the non-zero values of Y are large. The results in this section suggest—perhaps somewhat counterintuitively—that if one rescales the outcome such that the non-zero values are all large, the behavior of the ATE will be driven nearly entirely by the effect of the treatment on zero-valued outcomes and *not* by the distribution of the potential outcomes conditional on being positive. Moreover, the rescaling can be chosen to generate any magnitude for the ATE if the treatment affects the extensive margin.

2.3 Empirical illustrations from the *American Economic Review*

We illustrate the results in this section by evaluating the sensitivity to scaling of papers published in the *American Economic Review* (*AER*) using the $\operatorname{arcsinh}(Y)$ transformation. In November 2022, we used Google Scholar to search for “inverse hyperbolic sine” among papers published in the *AER* since 2018. Our search returned 17 papers that estimate treatment effects for an $\operatorname{arcsinh}$ -transformed outcome.⁷ Of these, 10 explicitly interpret the results as percents or elasticities, and 6 of the remaining 7 do not directly interpret the units. See [Appendix Table 1](#) for a list of the papers

⁷We consider papers with both binary and non-binary treatments, as our theoretical results extend easily to non-binary treatments; see [Remark 3](#).

and relevant quotes. Of the 17 total papers using $\text{arcsinh}(Y)$, 10 had publicly available replication data that allowed us to replicate the original estimates and assess their sensitivity to scaling.⁸ For our replications, we focus on the first specification using $\text{arcsinh}(Y)$ presented in a table in the paper, which we view as a reasonable proxy for the paper’s main specification using $\text{arcsinh}(Y)$.⁹

We assess the sensitivity of these results by re-running exactly the same procedure as in the original paper, except replacing $\text{arcsinh}(Y)$ with $\text{arcsinh}(100 \cdot Y)$. Thus, for example, if the original paper estimated a treatment effect for the arcsinh of an outcome measured in dollars, we use the same procedure to re-estimate the treatment effect for the arcsinh of the outcome measured in cents. Since Equation (2) shows that the sensitivity to scaling depends on the size of the extensive margin effect, we also estimate the extensive margin effect by using the same procedure as in the original paper but with the outcome $\mathbb{1}[Y > 0]$.

Paper	Treatment Effect Using:			Change from rescaling units:	
	$\text{arcsinh}(Y)$	$\text{arcsinh}(100 \cdot Y)$	Ext. Margin	Raw	%
Azoulay et al (2019)	0.003	0.017	0.003	0.014	464
Fetzer et al (2021)	-0.177	-0.451	-0.059	-0.273	154
Johnson (2020)	-0.179	-0.448	-0.057	-0.269	150
Carranza et al (2022)	0.201	0.453	0.055	0.252	125
Cao and Chen (2022)	0.038	0.082	0.010	0.044	117
Rogall (2021)	1.248	2.150	0.195	0.902	72
Moretti (2021)	0.053	0.066	0.000	0.013	24
Berkouwer and Dean (2022)	-0.498	-0.478	0.010	0.020	-4
Arora et al (2021)	0.113	0.110	-0.001	-0.003	-3
Hjort and Poulsen (2019)	0.354	0.354	0.000	0.000	0

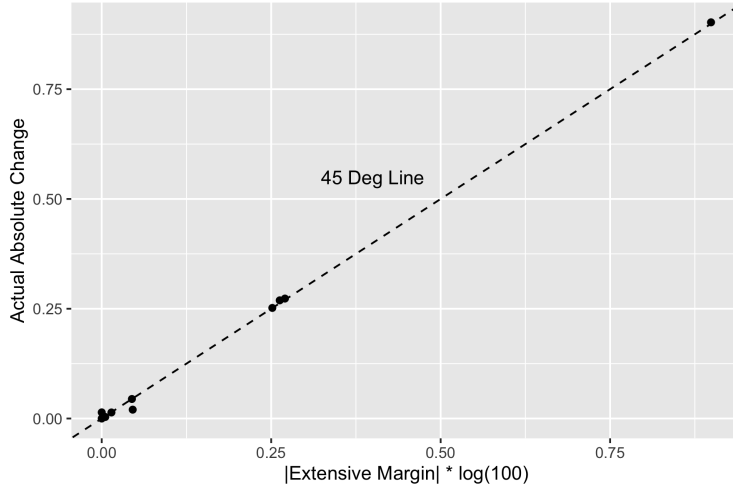
Table 2: Change in estimated treatment effects from re-scaling the outcome by a factor of 100 in papers published in the *AER* using $\text{arcsinh}(Y)$

The results of this exercise, shown in Table 2, illustrate that treatment effect estimates can be quite sensitive to the scaling of the outcome when the extensive margin is not near-zero. In 5 of the 10 replicable papers, multiplying the outcome by a factor of 100 changes the estimated treatment effect by more than 100% of the original estimate. Figure 1 shows that the change in the estimated treatment effect is larger when the extensive margin effect is larger, with the change lining up very closely with the approximation given in Equation (2).

⁸For two papers, there were slight discrepancies between our replication of the original result and the result reported in the paper, but these affected only the third decimal place.

⁹We use the first coefficient presented in a figure for one paper without any tables in the main text using $\text{arcsinh}(Y)$. If the first specification presented is a validation check (e.g. a pre-trends test), we use the first specification of causal interest.

Figure 1: Change from multiplying outcome by 100 versus extensive margin effect



Note: For each replicated paper, this figure shows the absolute value of the change in the estimated treatment effect from multiplying the outcome by 100 on the y -axis against $\log(100)$ times the absolute value of the extensive margin effect on the x -axis. If the approximation in Equation (2) were exact, all points would lie on the 45 degree line.

3 Sensitivity to scaling for other ATEs

Our results so far show that ATEs for transformations that are defined at zero and approximate $\log(y)$ are arbitrarily sensitive to scaling. What other options are available when there are zero-valued outcomes? To help delineate alternative options, in this section we provide a result showing what desirable properties a parameter defined with zero-valued outcomes can have. Specifically, we establish a “trilemma”: when there are zero-valued outcomes, there is no parameter that (a) takes the form $\theta_g = E_P[g(Y(1), Y(0))]$, (b) is scale-invariant, and (c) is point-identified. Any approach for settings with zero-valued potential outcomes must therefore abandon one of the properties (a)-(c); in Section 4 below we discuss several approaches that relax one (or more) of these requirements.

Before stating our formal result, we must make precise what we mean by scale-invariance and point-identification. We say that g is scale-invariant if its value is the same under any re-scaling of the units of y by a positive constant a .

Definition 1. We say that the function g is *scale-invariant* if it is homogeneous of degree zero, i.e. $g(y_1, y_0) = g(ay_1, ay_0)$ for all $a, y_1, y_0 > 0$.

To define point-identification, recall that P denotes the joint distribution of $(Y(1), Y(0))$, and $P_{Y(d)}$ denotes the marginal distribution of $Y(d)$. We consider the setting where the marginal distributions $P_{Y(1)}, P_{Y(0)}$ are identified from the data, but not the full joint distribution P , as in, e.g., Fan, Guerre and Zhu (2017). In a randomized controlled trial, for example, $P_{Y(d)}$ is identified from the distribution $Y \mid D = d$, but the joint distribution P is not identified since we never observe $Y(1)$ and $Y(0)$ simultaneously.¹⁰ We will thus say θ_g is point-identified if it depends only on the

¹⁰ $P_{Y(d)}$ is likewise identified under unconfoundedness and overlap assumptions in observational studies. In other

marginal distributions of the potential outcomes, and not on the joint distribution, which can never be learned directly from the data.

Definition 2 (Identification). We say that θ_g is *point-identified from the marginals at P* if for every joint distribution Q with the same marginals as P (i.e. such that $Q_{Y(d)} = P_{Y(d)}$ for $d = 0, 1$), $E_P[g(Y(1), Y(0))] = E_Q[g(Y(1), Y(0))]$. For a class of distributions \mathcal{P} , we say that θ_g is *point-identified over \mathcal{P}* if for every $P \in \mathcal{P}$, θ_g is point-identified from the marginals at P .

We will denote by \mathcal{P}_+ the set of distributions on $[0, \infty)^2$. Thus, θ_g is point-identified over \mathcal{P}_+ if it is always identified when Y takes on zero or weakly positive values. Our next result formalizes that it is not possible to have a parameter of the form $E_P[g(Y(1), Y(0))]$ that is both scale-invariant and point-identified over \mathcal{P}_+ .

Proposition 2 (A trilemma). *The following three properties cannot hold simultaneously:*

- (a) $\theta_g = E_P[g(Y(1), Y(0))]$ for a non-constant function $g : [0, \infty)^2 \rightarrow \mathbb{R}$ that is weakly increasing in its first argument.
- (b) The function g is scale-invariant.
- (c) θ_g is point-identified over \mathcal{P}_+ .

To prove [Proposition 2](#), we show that the only parameter satisfying properties (a) and (b) that is point-identified over distributions for which Y is *strictly* positively-valued is the ATE in logs (up to an affine transformation). Since $\log(0)$ is not well-defined, it follows that there are no parameters satisfying the three properties when one allows for zero-valued outcomes. Any parameter that is well-defined when there are zero-valued outcomes must therefore abandon at least one of properties (a)–(c).

4 Empirical approaches with zero-valued outcomes

Our theoretical results above imply that when there are zero-valued outcomes, the researcher cannot take a log-like transform of the outcome and interpret the resulting ATE as a percentage effect, since unlike a percentage, the ATE depends on the units of the outcome. In this section, we highlight some other parameters that are well-defined and easily interpreted when there are zero-valued outcomes. Of course, any alternative parameter must necessarily drop one of the requirements in the trilemma in [Proposition 2](#), but the choice of which to drop may depend on the researcher’s motivation.

common settings, the marginal distributions of the potential outcomes are identified for a subset of the population, e.g. for compliers in instrumental variables settings ([Abadie, 2003](#)) and for treated individuals in generalized DiD settings ([Athey and Imbens, 2006](#); [Roth and Sant’Anna, Forthcoming](#)). Note that under the standard parallel trends assumption, only the marginal *means* of the potential outcomes distributions are identified, an even weaker notion of identification.

To inform our discussion of alternative parameters, it is therefore useful to first enumerate several reasons why empirical researchers may contemplate estimating treatment effects for a log-transformed outcome in the first place, rather than just reporting an estimate of the ATE in levels:¹¹

(i) The researcher is interested in reporting an ATE with easily-interpretable units, such as “percents.”

(ii) The researcher believes that there are decreasing returns to the outcome, and thus wants to place more weight on treatment effects for individuals with low initial outcomes. For instance, the researcher may perceive it to be more meaningful to raise income from $Y(0) = \$10,000$ to $Y(1) = \$20,000$ than from $Y(0) = \$100,000$ to $Y(1) = \$110,000$, yet both of these treatment effects contribute equally to the ATE in levels.

(iii) The researcher is interested in both the intensive and extensive margin effects of the treatment, and is using the ATE for a log-like transformation as an approximation to the proportional effect along the intensive margin.

These motivations for targeting a parameter other than the ATE in levels will often be most salient in settings where the outcome variable is skewed. The ATE in levels may be difficult to interpret when Y spans several orders of magnitude—e.g., a change in employment of 1,000 is more meaningful in a small town than in New York City—thus motivating attention to a parameter with more interpretable units. Likewise, capturing diminishing marginal utility or understanding the intensive margin effect will typically be more important when there is substantial variation in the positive values of the outcome.

These three motivations also suggest different ways of breaking out of the trilemma in [Proposition 2](#). If the goal is to achieve a percentage interpretation, then one can consider scale-invariant parameters outside of the class $E_P[g(Y(1), Y(0))]$, such as the ATE in levels expressed as a percentage of the control mean, or the ATE for a normalized parameter \tilde{Y} that already has a percentage interpretation. Alternatively, if the goal is to capture concave social preferences over the outcome, then it is natural to specify how much we value the intensive margin relative to the extensive margin—thus abandoning scale-invariance. Finally, if the goal is to separately understand the intensive margin effect, the researcher can abandon point-identification (from the marginal distributions) and directly target the partially identified parameter $E[\log(Y(1)) - \log(Y(0)) \mid Y(0) > 0, Y(1) > 0]$, the effect in logs for individuals with positive outcomes under both treatments. We address each of these cases in turn, with a summary in [Table 3](#).

¹¹We focus on reasons why the researcher may be interested in parameters other than the ATE in levels. In some settings, the researcher may actually be interested in the ATE in levels, but may choose to estimate it by first estimating the ATE in logs (or a related transformation) and then re-scaling the estimate by the control mean. This can potentially improve precision in settings where the outcome has a long right tail by reducing sensitivity to outliers. We note, however, that this type of procedure only recovers the ATE in levels under additional assumptions on the joint distribution of the potential outcomes (e.g. constant proportional treatment effects), and such assumptions for one transformation may be incompatible with those for another. See [Wooldridge \(1992\)](#) for related discussion.

Description	Parameter	Main property sacrificed?	Pros/Cons
Normalized ATE	$E[Y(1) - Y(0)]/E[Y(0)]$	$E[g(Y(1), Y(0))]$	<i>Pro:</i> Percent interpretation <i>Con:</i> Does not capture decreasing returns
Normalized outcome	$E[Y(1)/X - Y(0)/X]$	$E[g(Y(1), Y(0))]$	<i>Pro:</i> Per-unit- X interpretation <i>Con:</i> Need to find sensible X
Explicit tradeoff of intensive/extensive margins	ATE for $m(y) = \begin{cases} \log(y) & y > 0 \\ -x & y = 0 \end{cases}$	Scale-invariance	<i>Pro:</i> Explicit tradeoff of two margins <i>Con:</i> Need to choose x ; Monotone only if support excludes $(0, e^{-x})$
Intensive margin effect	$E\left[\log\left(\frac{Y(1)}{Y(0)}\right) \mid Y(1) > 0, Y(0) > 0\right]$	Point-identification	<i>Pro:</i> ATE in logs for the intensive margin <i>Con:</i> Partial identification

Table 3: Summary of alternative target parameters

4.1 When the goal is interpretable units

We first consider the case where the researcher’s primary goal is to obtain a treatment effect parameter with easily-interpretable units, such as percentages.

Normalizing the ATE in levels. One possibility is to target the parameter

$$\theta_{\text{ATE}\%} = \frac{E[Y(1) - Y(0)]}{E[Y(0)]},$$

which is the ATE *in levels* expressed as a *percentage of the control mean*. For example, if a researcher is studying a program D meant to reduce healthcare spending Y , then $\theta_{\text{ATE}\%}$ is the percentage reduction in costs from implementing the program. This parameter is point-identified and scale-invariant, and thus has an intuitive percentage interpretation. It is not an average of individual-level effects, however—i.e., it does not take the form $E_P[g(Y(1), Y(0))]$ —and thus avoids the trilemma in [Proposition 2](#). We note that $\theta_{\text{ATE}\%}$ is consistently estimable by Poisson regression (see Chapter 18.2 in [Wooldridge, 2010](#)) under the potential outcomes model and an appropriate identifying assumption (e.g. unconfoundedness).¹²

We emphasize, however, that the numerator of $\theta_{\text{ATE}\%}$ is the ATE in levels. Thus, $\theta_{\text{ATE}\%}$ may be hard to interpret in settings where the ATE in levels is itself hard to interpret. For one, it may be dominated by a small number of individuals in the right-tail of the outcome distribution. This may be undesirable in some contexts: a researcher studying an anti-poverty program meant to increase the earnings for poor individuals, for example, may not want the parameter to be dominated by a small subset of the population with very large earnings. We next turn to alternative approaches that may be more appropriate in settings where Y is highly-skewed, such that interpreting the ATE

¹²With a randomly assigned D , for example, Poisson regression using the pseudo-likelihood $Y \mid D \sim \text{Pois}(e^{\alpha + \beta D})$ estimates the population coefficient $e^\beta = E[Y(1)]/E[Y(0)] = \theta_{\text{ATE}\%} + 1$.

in levels (expressed as a percentage) may be difficult.

Normalizing the outcome. A second, related approach to obtaining a treatment effect with more intuitive units is to estimate the ATE for a transformed outcome of the form $\tilde{Y} = Y/X$, where Y is the original outcome and X is some pre-determined characteristic. For example, consider a setting where Y is employment in a particular area. The treatment effect in levels for Y may be difficult to interpret, since a change in employment of 1,000 means something very different in New York City versus a small rural town. However, if X is the area’s population, then \tilde{Y} is the employment-to-population ratio, which may be more comparable across places, and is already in percentage (i.e. per capita) units. We note that the ATE for \tilde{Y} is a scale-invariant, point-identified parameter of the form $\theta = E_P[g(Y(1), Y(0), X)]$, and thus escapes the trilemma in [Proposition 2](#) by avoiding property (a).¹³ The viability of this approach, of course, depends on having a variable X such that the normalized outcome \tilde{Y} is of economic interest. We suspect that in many contexts, reasonable options will be available, including pre-treatment observations of the outcome (assuming these are positive), or the *predicted* control outcome given some observable characteristics ($X = E[Y(0) | W]$, for observable characteristics W).

4.2 When the goal is to capture decreasing returns

We next consider the case where the researcher wants to capture some form of decreasing marginal utility over the outcome. For example, when Y is strictly positively valued, the ATE in logs corresponds with the change in utility from implementing the treatment for a utilitarian social planner with log utility over the outcome, $U = E[\log(Y)]$. Intuitively, this social welfare function captures the fact that the planner values a percentage point change in the outcome equally for all individuals, regardless of their initial level of the outcome.

Of course, log utility is not well-defined when there is an extensive margin. Thus, a planner with log utility over the intensive margin must specify how much they value the intensive versus extensive margins. Recall from [Section 2.1](#) that when using transformations like $\log(1 + y)$ or $\operatorname{arcsinh}(y)$, the scaling of the outcome implicitly determines the weights placed on these margins.

Instead of implicitly weighting the margins via the scaling of Y , one can directly specify how much they value each margin of treatment. If one has log utility over changes in, say, earnings along the intensive margin, then a natural approach is to specify how much one values a change in earnings from 0 to 1 relative to a percentage change in earnings for those with non-zero earnings. If, for example, one values the extensive margin effect of moving from 0 to 1 the same as an x percent increase in earnings, then one might consider setting $m(y) = \log(y)$ for $y > 0$ and $m(0) = -x$. The ATE for this transformation can be interpreted as an approximate percentage (log point) effect, where an increase from 0 to 1 is valued at x log points.¹⁴

¹³It is scale-invariant in the sense that $g(y_1, y_0, x) = g(ay_1, ay_0, ax)$.

¹⁴Note that this transformation will generally only be sensible if the support of Y excludes $(0, e^{-x})$, since otherwise the function $m(y)$ is not monotone in y over the support of Y . It is common, however, to have a lower-bound on non-zero values of the outcome; e.g., a firm cannot have between 0 and 1 employees.

We emphasize that for a fixed value of x , this approach necessarily depends on the scaling of the outcome (thus avoiding the trilemma in [Proposition 2](#)). However, this may not be so concerning since the appropriate choice of x also depends on the units of the outcome—e.g. saying a change from 0 to 1 is worth x percent means something very different if 1 corresponds with one dollar versus a million dollars. In other words, ATEs for transformations such as $\text{arcsinh}(Y)$ may be difficult to interpret because the scaling of the outcome implicitly determines the relative importance of the intensive and extensive margins; this approach avoids that difficulty by *explicitly* taking a stand on the tradeoff between these two margins. Nevertheless, a challenge with this approach is that researchers may have differing opinions over the appropriate choice of x (or more generally, over the appropriate social welfare function).

4.3 When the goal is to understand intensive and extensive margins

Finally, we consider the case where the researcher is interested in understanding the intensive and extensive margin effects, separately. A common question in the literature on job training programs ([Card, Kluge and Weber, 2010](#)), for instance, is whether a program raises participants’ earnings by helping them find a job—which would be expected only to have an extensive-margin effect—or by increasing human capital, which would be expected to also affect the intensive margin. In such settings, it is natural to target separate parameters for the intensive and extensive margins. For example, the parameter

$$\theta_{\text{Intensive}} = E[\log(Y(1)) - \log(Y(0)) \mid Y(1) > 0, Y(0) > 0]$$

captures the ATE in logs for those who would have a positive outcome regardless of their treatment status. The parameter $\theta_{\text{Intensive}}$ is scale-invariant but is not point-identified from the marginal distributions of the potential outcomes (thus avoiding the trilemma in [Proposition 2](#)), and therefore cannot be consistently estimated without further assumptions.¹⁵ However, [Lee \(2009\)](#) popularized a method for obtaining bounds on $\theta_{\text{Intensive}}$ under the monotonicity assumption that, for example, everyone with positive earnings without receiving a training would also have positive earnings when receiving the training.¹⁶ Bounds on $\theta_{\text{Intensive}}$ can be reported alongside measures of the extensive margin effect, such as the change in the probability of having a non-zero outcome, $P(Y(1) > 0) - P(Y(0) > 0)$, and the average (log) outcome for individuals for whom treatment affects the extensive margin, $E[\log(Y(1)) \mid Y(1) > 0, Y(0) = 0]$, which can be bounded similarly to $\theta_{\text{Intensive}}$.¹⁷ One can potentially tighten the bounds (or restore point-identification) by imposing additional assumptions

¹⁵ $\theta_{\text{Intensive}}$ also does not take the form $E_P[g(Y(1), Y(0))]$, although it can be written as

$$\frac{E_P[\mathbb{1}[Y(1) > 0, Y(0) > 0] \log(Y(1)/Y(0))]}{E_P[\mathbb{1}[Y(1) > 0, Y(0) > 0]}}$$

where both the numerator and denominator take this form.

¹⁶See, also, [Zhang and Rubin \(2003\)](#) for related results, including bounds without the monotonicity assumption.

¹⁷Under the monotonicity assumption imposed by [Lee \(2009\)](#),

$$E[\log(Y(1)) \mid Y(1) > 0] = (1 - \theta)E[\log(Y(1)) \mid Y(1) > 0, Y(0) > 0] + \theta E[\log(Y(1)) \mid Y(1) > 0, Y(0) = 0],$$

on the joint distribution of the potential outcomes—e.g. by assuming that $Y(1)$ stochastically dominates $Y(0)$ (Zhang et al., 2009).

We note that the parameter $\theta_{\text{Intensive}}$ is generally distinct from the “intensive margin” marginal effects implied by two-part models (2PMs), which were recommended for scenarios with zero-valued outcomes by Mullahy and Norton (2022), among others. In Appendix E, we consider the causal interpretation of the marginal effects of 2PMs, building on the discussion in Angrist (2001). Our decomposition shows that the marginal effects from 2PMs yield the sum of a causal parameter similar to $\theta_{\text{Intensive}}$ as well as a “selection term” comparing potential outcomes for individuals for whom treatment only has an intensive margin effect to those with an extensive margin effect. It thus may be difficult to ascribe a causal interpretation to the marginal effects of 2PMs without assumptions about this selection.

5 Conclusion

It is common in empirical work to estimate ATEs for transformations such as $\log(1+Y)$ or $\text{arcsinh}(Y)$ which are well-defined at zero and behave like $\log(Y)$ for large values of Y . We show that the ATEs for such transformations cannot be interpreted as percentages, since they depend arbitrarily on the units of the outcome. Further, we show that any parameter of the form $\theta_g = E_P[g(Y(1), Y(0))]$ must be scale-dependent if it is point-identified and well-defined at zero. We discuss several alternative approaches, including estimating ATEs for normalized outcomes, explicitly calibrating the value placed on the intensive versus extensive margins, and separately estimating effects for the intensive and extensive margins.

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where $\theta = P(Y(0) > 0)/P(Y(1) > 0)$. Since $E[\log(Y(1)) | Y(1) > 0]$ and θ are identified (from the marginal distributions of the potential outcomes), bounds on $E[\log(Y(1)) | Y(1) > 0, Y(0) > 0]$ as derived in Lee (2009) immediately imply bounds on $E[\log(Y(1)) | Y(1) > 0, Y(0) = 0]$.

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A Proofs for Section 2 (Sensitivity to scaling for transformations that behave like $\log(Y)$)

Proposition 1. *Suppose that:*

1. (The function m is continuous and increasing) $m : [0, \infty) \rightarrow \mathbb{R}$ is a continuous, weakly increasing function.
2. (The function m behaves like \log for large values) $m(y)/\log(y) \rightarrow 1$ as $y \rightarrow \infty$.
3. (Treatment affects the extensive margin) $P(Y(1) = 0) \neq P(Y(0) = 0)$.
4. (Finite expectations) $E_{P_{Y(d)}}[|\log(Y(d))| \mid Y(d) > 0] < \infty$ for $d = 0, 1$.¹⁸

Then, for every $\theta^* \in (0, \infty)$, there exists an $a > 0$ such that $|\theta(a)| = \theta^*$. In particular, $\theta(a)$ is continuous with $\theta(a) \rightarrow 0$ as $a \rightarrow 0$ and $|\theta(a)| \rightarrow \infty$ as $a \rightarrow \infty$.

Proof. Note that $\theta(0) = E_P[m(0)] - E_P[m(0)] = 0$. Additionally, [Proposition 4](#) below implies that $|\theta(a)| \rightarrow \infty$ as $a \rightarrow \infty$. To establish the proof, it thus suffices to show that $\theta(a)$ is continuous on $[0, \infty)$. The desired result is then immediate from the intermediate value theorem.

To establish continuity, fix some $a \in [0, \infty)$ and consider a sequence $a_n \rightarrow a$. Without loss of generality, assume $a_n < a + 1$ for all n . Let $m_{a_n}(y) = m(a_n y)$. Since m is continuous, $m_{a_n}(y) \rightarrow m_a(y)$ pointwise. Since $m(y)/\log(y) \rightarrow 1$ as $y \rightarrow \infty$, there exists \bar{y} such that $m(y) < 2\log(y)$ for all $y \geq \bar{y}$. From the monotonicity of m , it follows that

$$m(0) \leq m(y) \leq \mathbb{1}[y \leq \bar{y}]m(\bar{y}) + \mathbb{1}[y > \bar{y}]2\log(y)$$

and hence

$$\begin{aligned} m(0) \leq m_{a_n}(y) &\leq \mathbb{1}[a_n y \leq \bar{y}]m(\bar{y}) + \mathbb{1}[a_n y > \bar{y}]2\log(a_n y) \\ &\leq |m(\bar{y})| + 2 \cdot \mathbb{1}[y > 0] \cdot (|\log(a + 1)| + |\log(y)|) =: \bar{m}(y). \end{aligned}$$

for all n . Hence, we have that $|m_{a_n}(y)| \leq |m(0)| + |\bar{m}(y)|$ for all n , and the bounding function is integrable for $Y(d)$ for $d = 0, 1$ by the fourth assumption of the proposition. It follows from the dominated convergence theorem that $E_P[m_{a_n}(Y(d))] \rightarrow E_P[m_a(Y(d))]$ for $d = 0, 1$, and thus $\theta(a_n) \rightarrow \theta(a)$, as we wished to show. \square

B Proofs for Section 3 (Sensitivity to scaling for other ATEs)

Proposition 2 (A trilemma). *The following three properties cannot hold simultaneously:*

- (a) $\theta_g = E_P[g(Y(1), Y(0))]$ for a non-constant function $g : [0, \infty)^2 \rightarrow \mathbb{R}$ that is weakly increasing in its first argument.

¹⁸This assumption simply ensures that $E_{P_{Y(d)}}[|m(aY(d))| \mid Y > 0]$ exists for all values of $a > 0$.

(b) The function g is scale-invariant.

(c) θ_g is point-identified over \mathcal{P}_+ .

Proof. To establish the proof of Proposition 2, we first prove the following result, which shows that the only scale-invariant parameter of the form $E_P[g(Y(1), Y(0))]$ that is identified over distributions on the positive reals is the ATE in logs (up to an affine transformation).

Proposition 3. *Let \mathcal{P}_{++} denote the set of distributions over compact subsets of $(0, \infty)^2$. Suppose $g : (0, \infty)^2 \rightarrow \mathbb{R}$ is weakly increasing in y_1 and scale-invariant. Then θ_g is point-identified over \mathcal{P}_{++} if and only if $g = c \cdot (\log(y_1) - \log(y_0)) + d$, for constants $c \geq 0$ and $d \in \mathbb{R}$.*

Proof. We first show that point-identification over \mathcal{P}_{++} implies that $g(\cdot, \cdot)$ must be additively separable. We do so by considering the points $\{y_0, y_0 + b\} \times \{y_1, y_1 + a\}$ on a rectangular grid. If $g(\cdot, \cdot)$ is not additively separable, then its expectation with respect to distributions supported on the rectangular grid depends on the correlation. Similar arguments appear in, e.g., Fan et al. (2017).

Formally, suppose that there exist positive values $y_1, y_0, a, b > 0$ such that

$$g(y_1, y_0) + g(y_1 + a, y_0 + b) \neq g(y_1 + a, y_0) + g(y_1, y_0 + b).$$

Now, consider the marginal distributions $P_{Y(d)}$ such that $P(Y(1) = y_1) = \frac{1}{2} = P(Y(1) = y_1 + a)$ and $P(Y(0) = y_0) = \frac{1}{2} = P(Y(0) = y_0 + b)$. Let P_1 and P_2 denote the joint distributions corresponding with these marginals and perfect positive and negative correlation of the potential outcomes, respectively. Then we have that

$$\begin{aligned} E_{P_1}(g(Y(1), Y(0))) &= \frac{1}{2} (g(y_1, y_0) + g(y_1 + a, y_0 + b)) \\ &\neq \frac{1}{2} (g(y_1 + a, y_0) + g(y_1, y_0 + b)) \\ &= E_{P_2}(g(Y(1), Y(0))), \end{aligned}$$

and thus θ_g is not point-identified from the marginals at P_1 . Hence, if θ_g is identified over \mathcal{P}_{++} , then it must be that

$$g(y_1, y_0) + g(y_1 + a, y_0 + b) = g(y_1 + a, y_0) + g(y_1, y_0 + b) \text{ for all } y_1, y_0, a, b > 0,$$

and hence

$$g(y_1 + a, y_0) - g(y_1, y_0) = g(y_1 + a, y_0 + b) - g(y_1, y_0 + b) \text{ for all } y_1, y_0, a, b > 0.$$

It follows that we can write $g(y_1, y_0) = r(y_1) + q(\frac{1}{y_0})$, where $r(y_1) = g(y_1, 1) - g(1, 1)$ and $q(\frac{1}{y_0}) = g(1, y_0)$.

Second, we show that homogeneity of degree zero, combined with monotonicity, implies that g

must be a difference in logarithms. Observe that since g is scale-invariant,

$$g(y_1, y_0) = g\left(\frac{y_1}{y_0}, \frac{y_0}{y_0}\right) = g\left(\frac{y_1}{y_0}, 1\right) =: h\left(\frac{y_1}{y_0}\right),$$

where h is an increasing function. We thus have that for any $a, b > 0$,

$$\begin{aligned} g(1, 1) &= h(1) = r(1) + q(1) \\ g(a, 1) &= h(a) = r(a) + q(1) \\ g\left(1, \frac{1}{b}\right) &= h(b) = r(1) + q(b) \\ g\left(a, \frac{1}{b}\right) &= h(ab) = r(a) + q(b) \end{aligned}$$

and hence $h(ab) = h(a) + h(b) - h(1)$. It follows that $\tilde{h}(x) = h(x) - h(1)$ is an increasing function such that $\tilde{h}(ab) = \tilde{h}(a) + \tilde{h}(b)$ for all $a, b \in \mathbb{R}$, i.e. an increasing function satisfying Cauchy's logarithmic function equation: $\phi(ab) = \phi(a) + \phi(b)$ for all positive reals a, b . Recall that if a function is increasing, then it has countably many discontinuity points, and thus is continuous somewhere. It is a well-known result in functional equations that the only solutions to Cauchy's logarithmic equation are of the form $\phi(t) = c \log(t)$, if we require that these solutions are continuous at some point; see [Aczél \(1966\)](#), Theorem 2 in Section 2.1.2.¹⁹ Since we require monotonicity, the constant $c \geq 0$. Thus, $g(y_1, y_0) = h(y_1/y_0) = \tilde{h}(y_1/y_0) + \tilde{h}(1) = c \log(y_1) - c \log(y_0) + \tilde{h}(1)$. Letting $d = \tilde{h}(1)$ completes the proof of [Proposition 3](#). \square

Note that if $g : [0, \infty)^2 \rightarrow \mathbb{R}$ is increasing in y_1 , then it cannot be equal to $c \log(y_1/y_0) + d$ for $c > 0$ everywhere on $(0, \infty)^2$, since this would imply that $\lim_{y_1 \rightarrow 0} g(y_1, 1) = -\infty < g(0, 1)$. The proof of [Proposition 2](#) is then immediate from [Proposition 3](#), which shows that if properties (a) and (b) are satisfied, and θ_g is point-identified over $\mathcal{P}_{++} \subset \mathcal{P}_+$, then $g = c \log(y_1/y_0) + d$ on $(0, \infty)^2$. \square

C Extensions

C.1 Sensitivity to finite changes in scale

The following result formalizes the discussion in [Remark 2](#) about how the ATE for $m(Y)$ changes with finite changes in the scale of Y .

Proposition 4. *Under the conditions of [Proposition 1](#),²⁰ as $a \rightarrow \infty$,*

$$E_P[m(a \cdot Y(1)) - m(a \cdot Y(0))] = (P(Y(1) > 0) - P(Y(0) > 0)) \cdot \log(a) + o(\log(a)).$$

¹⁹Correspondingly, non-trivial solutions to Cauchy's logarithmic equations are highly ill-behaved.

²⁰Continuity of m is not needed for this result.

Proof. Fix a sequence $a_n \rightarrow \infty$, and without loss of generality, assume $a_n > e$. We will show that

$$\frac{1}{\log a_n} E_P[m(a_n Y(1)) - m(a_n Y(0))] \rightarrow P(Y(1) = 0) - P(Y(0) = 0). \quad (3)$$

Define $f_n(y) = m(a_n y)/\log(a_n)$. Note that $f_n(y) \rightarrow \mathbb{1}[y > 0]$ pointwise, since $f_n(0) = m(0)/\log(a_n) \rightarrow 0$, while for $y > 0$,

$$f_n(y) = \frac{m(a_n y)}{\log(a_n)} = \frac{m(a_n y)}{\log(a_n y)} \frac{\log(a_n) + \log(y)}{\log(a_n)} \rightarrow 1,$$

where we use the fact that $m(y)/\log(y) \rightarrow 1$ as $y \rightarrow \infty$ by assumption. We showed in the proof to [Proposition 1](#) that

$$|m(y)| \leq \kappa + 2 \cdot \mathbb{1}[y > 0] \cdot |\log(y)|$$

where κ is a constant not depending on y . It follows that

$$|f_n(y)| = \frac{|m(a_n y)|}{\log(a_n)} \leq \kappa + 2 \cdot \mathbb{1}[y > 0] \cdot (1 + |\log(y)|).$$

Further, since $E_P[|\log(Y(d))| \mid Y(d) > 0]$ is finite by assumption, the upper bound is integrable for $y = Y(d)$ for $d = 0, 1$. It follows from the dominated convergence theorem that

$$E_P[f_n(Y(d))] = E_P\left[\frac{m(a_n Y(d))}{\log(a_n)}\right] \rightarrow E_P[\mathbb{1}[Y(d) > 0]] = P(Y(d) > 0).$$

[Equation \(3\)](#) follows immediately from the continuous mapping theorem, which completes the proof. \square

C.2 Extension to continuous treatments

Although we focus on binary treatment in the main text for simplicity, similar issues arise with continuously distributed D . Suppose now that D can take a continuum of values on some set $\mathcal{D} \subseteq \mathbb{R}$. Let $Y(d)$ denote the potential outcome at the dose d , and P the distribution of $Y(\cdot)$. Consider the parameter

$$\theta(a) = \int_{\mathcal{D}} \omega(d) E_P[m(aY(d))],$$

which is a weighted sum of the average values of $m(aY(d))$ across different values of d with weights $\omega(d)$. For example, in an RCT with a continuous treatment, a regression of $m(aY)$ on D yields a parameter of the form $\theta(a)$ where, by the Frisch-Waugh-Lovell theorem, the weights are proportional to $(d - E[D])p(d)$ and integrate to 0.²¹

We now show that $\theta(a)$ can be made to have arbitrary magnitude via the choice of a when there is an extensive margin effect. In particular, by an extensive margin effect we mean that $\int \omega(d) P(Y(d) > 0) \neq 0$, i.e. when there is an average effect on the probability of a zero outcome,

²¹Here, $p(d)$ denotes the density of D at d over the randomization distribution.

using the same weights $\omega(d)$ that are used for $\theta(a)$. When $\theta(a)$ is the regression of $m(aY)$ on D in an RCT, for example, $\int \omega(d)P(Y(d) > 0) \neq 0$ if the regression of $\mathbb{1}[Y > 0]$ on D yields a non-zero coefficient.

Proposition 5. *Suppose that:*

1. *The function m satisfies parts 1 and 2 of Proposition 1.*
2. *(Extensive margin effect) $\int_{\mathcal{D}} \omega(d)P(Y(d) > 0) \neq 0$.*
3. *(Bounded expectations) For all d , $E_P[|\log(Y(d))| \mid Y(d) > 0] < \infty$.*
4. *(Regularity for weights) The weights $\omega(d)$ satisfy $\int_{\mathcal{D}} \omega(d) = 0$, $\int_{\mathcal{D}} |\omega(d)| < \infty$ and $\int_{\mathcal{D}} |\omega(d)| \cdot E_P[|\log(Y(d))| \mid Y(d) > 0] < \infty$.*

Then for every $\theta^ \in (0, \infty)$, there exists $a > 0$ such $|\theta(a)| = \theta^*$. In particular, $\theta(a)$ is continuous and $\theta(a) \rightarrow 0$ as $a \rightarrow 0$ and $|\theta(a)| \rightarrow \infty$ as $a \rightarrow \infty$.*

Proof. Note that $\theta(0) = \int \omega(d)m(0) = 0$. It thus suffices to show that $\theta(a)$ is continuous for $a \in [0, \infty)$ and that $|\theta(a)| \rightarrow \infty$ as $a \rightarrow \infty$. The result then follows from the intermediate value theorem.

We first show continuity. Fix $a \in [0, \infty)$ and a sequence $a_n \rightarrow a$. Let $f_n(d) = \omega(d)E_P[m(a_n Y(d))]$. We showed in the proof to Proposition 1 that $E_P[m(a_n Y(d))] \rightarrow E_P[m(aY(d))]$, and thus $f_n(d) \rightarrow \omega(d)E_P[m(aY(d))]$ pointwise. We also showed in the proof to Proposition 1 that for a_n sufficiently close to a ,

$$|m(a_n Y)| \leq \kappa + 2 \cdot \mathbb{1}[y > 0] \cdot |\log(y)|,$$

for a constant κ not depending on n . It follows that

$$|f_n(d)| \leq |\omega(d)| \cdot |\kappa| + 2|\omega(d)| \cdot E_P[|\log(Y(d))| \mid Y(d) > 0],$$

and the upper bound is integrable by part 4 of the Proposition. Hence, by the dominated convergence theorem, we have that $\theta(a_n) = \int_{\mathcal{D}} f_n(d) \rightarrow \int_{\mathcal{D}} \omega(d)E_P[m(aY(d))] = \theta(a)$, as needed.

To show that $|\theta(a)| \rightarrow \infty$ as $a \rightarrow \infty$, we will show that

$$\frac{\theta(a)}{\log(a)} \rightarrow \int_{\mathcal{D}} \omega(d)P[Y(d) > 0]$$

as $a \rightarrow \infty$. Consider $a_n \rightarrow \infty$, and suppose without loss of generality that $a_n > e$. Observe that

$$\frac{\theta(a_n)}{\log(a_n)} = \int_{\mathcal{D}} \omega(d) \frac{E_P[m(a_n Y(d))]}{\log(a_n)}.$$

We showed in the proof to Proposition 4 that for each d ,

$$\frac{E_P[m(a_n Y(d))]}{\log(a_n)} \rightarrow P(Y(d) > 0).$$

Letting $f_n(d) = \omega(d) \frac{E_P[m(a_n Y(d))]}{\log(a_n)}$, we thus have that $f_n(d) \rightarrow \omega(d)P(Y(d) > 0)$ pointwise. Moreover, we showed in the proof to [Proposition 1](#) that

$$|m(y)| \leq \kappa + 2 \cdot \mathbb{1}[y > 0] \cdot |\log(y)|$$

where κ is a constant not depending on y . It follows that

$$\frac{|m(a_n y)|}{\log(a_n)} \leq \kappa + 2 \cdot \mathbb{1}[y > 0] \cdot (1 + |\log(y)|)$$

and thus that

$$|f_n(d)| \leq |\omega(d)| \cdot (\kappa + 2 + 2E_P[|\log(Y(d))| \mid Y(d) > 0])$$

where the upper bound is integrable by the fourth part of the proposition. The result then follows from dominated convergence. □

C.3 Extension to OLS estimands

As noted in [Remark 4](#), our results imply that any consistent estimator of the ATE for an outcome of the form $m(aY)$ will be (asymptotically) sensitive to scaling when there is an extensive margin effect. Our results thus cover the OLS estimand when it is consistent (e.g. in an RCT or under unconfoundedness). Given the prominence of OLS in applied work—and the fact that it is sometimes used for non-causal analyses—we now provide a direct result on the sensitivity to scaling of the estimand of an OLS regression of an outcome of the form $m(aY)$ on an arbitrary random variable X .

Specifically, suppose that $(X, Y) \sim Q$, for $Y \in [0, \infty)$ and $X \in \mathbb{R}^J$, where the first element of X is a constant. Consider the OLS estimand

$$\beta(a) = E_Q[XX']^{-1} E_Q[Xm(aY)].$$

We assume that $E_Q[XX']$ is full-rank so that $\beta(a)$ is well-defined. Letting $\beta_j(a) = e'_j \beta(a)$ be the j^{th} element of $\beta(a)$, we will show that $\beta_j(a)$ can be made to have arbitrary magnitude via the choice of a if $\gamma_j \neq 0$, where

$$\gamma = E_Q[XX']^{-1} E_Q[X \mathbb{1}[Y > 0]]$$

is the coefficient from a regression of $\mathbb{1}[Y > 0]$ on X .

Proposition 6. *Suppose that*

1. *The function m satisfies parts 1 and 2 of [Proposition 1](#).*
2. *(Finite expectations) $E_Q[\|X\|] < \infty$ and $E_Q[\|X \log(Y)\| \mid Y > 0] < \infty$.*
3. *For some $j \in \{2, \dots, J\}$, $\gamma_j \neq 0$.*

Then for every $\beta_j \in (0, \infty)$, there exists $a > 0$ such that $|\beta_j(a)| = \beta_j$. In particular $\beta_j(a)$ is continuous with $\beta_j(a) \rightarrow 0$ as $a \rightarrow 0$ and $|\beta_j(a)| \rightarrow \infty$ as $a \rightarrow \infty$. Moreover, $\beta_j(a)/\log(a) \rightarrow \gamma_j$ as $a \rightarrow \infty$.

We note that [Proposition 6](#) implies that the OLS estimator for the j^{th} coefficient, $\hat{\beta}_j(a)$, will be arbitrarily sensitive to the choice of a when the corresponding extensive margin OLS estimator $\hat{\gamma}_j$, is non-zero. This follows immediately from setting Q to be the empirical distribution of $(Y_i, X_i)_{i=1}^N$ and applying [Proposition 6](#) (note that part 2 of the Proposition is trivially satisfied for the empirical distribution, since X and Y are both bounded over the empirical distribution).

Proof. Note that $\beta(0) = E_Q[XX']^{-1}E[Xm(0)]$, is the coefficient from a regression of a constant outcome $m(0)$ on X , and thus $\beta_1(0) = m(0)$ while $\beta_k(0) = 0$ for $k \geq 2$. Thus $\beta_j(0) = 0$. To complete the proof, we will show that $|\beta_j(a)| \rightarrow \infty$ as $a \rightarrow \infty$ and that $\beta_j(a)$ is continuous for $a \in [0, \infty)$. The result then follows from the intermediate value theorem.

For ease of notation, let $\nu' = e'_j E_Q[XX']^{-1}$, so that $\beta_j(a) = E_Q[\nu' X m(aY)]$.

We first show that $\beta_j(a) \rightarrow \infty$ as a diverges. Consider a sequence $a_n \rightarrow \infty$, and assume without loss of generality that $a_n > e$. Let $f_n(x, y) = \nu' x \cdot m(a_n y)/\log(a_n)$. Observe that $f_n(x, y) \rightarrow \nu' x \cdot \mathbb{1}[y > 0]$ pointwise, since $f_n(x, 0) = \nu' x \cdot m(0)/\log(a_n) \rightarrow 0$, while for $y > 0$,

$$f_n(x, y) = \nu' x \cdot \frac{m(a_n y)}{\log(a_n)} = \nu' x \cdot \frac{m(a_n y)}{\log(a_n y)} \frac{\log(a_n) + \log(y)}{\log(a_n)} \rightarrow \nu' x,$$

where we use the fact that $m(y)/\log(y) \rightarrow 1$ as $y \rightarrow \infty$. We showed in the proof to [Proposition 4](#) that

$$\frac{|m(a_n y)|}{\log(a_n)} \leq \kappa + 2 \cdot \mathbb{1}[y > 0] \cdot (1 + |\log(y)|),$$

which implies that

$$|f_n(x, y)| \leq |\nu' x \cdot (\kappa + 2 \cdot \mathbb{1}[y > 0] \cdot (1 + |\log(y)|))| =: \bar{f}(x, y).$$

Moreover, part 2 of the proposition implies that $\bar{f}(X, Y)$ is integrable. From the dominated convergence theorem, it follows that

$$\frac{\beta_j(a_n)}{\log(a_n)} = E_Q[f_n(X, Y)] \rightarrow E_Q[\nu' X \mathbb{1}[Y > 0]] = \gamma_j \neq 0.$$

Hence, we see that $\beta_j(a_n) = \gamma_j \log(a_n) + o(\log(a_n))$, and thus $|\beta_j(a_n)| \rightarrow \infty$.

To complete the proof, we show continuity of $\beta_j(a)$. Fix $a \in [0, \infty)$, and consider a sequence $a_n \rightarrow a$. Assume without loss of generality that $a_n < a + 1$ for all n . Let $f_n(x, y) = \nu' x \cdot m(a_n y)$. From the continuity of m , we have that $f_n(x, y) \rightarrow \nu' x \cdot m(a y)$ pointwise. We showed in the proof to [Proposition 1](#) that there exists some κ (not depending on n) such that

$$|m(a_n y)| \leq \kappa + 2\mathbb{1}[y > 0] \cdot |\log(y)|.$$

Hence,

$$|f_n(x, y)| \leq |\nu'x \cdot (\kappa + 2\mathbb{1}[y > 0]) \log(y)|.$$

Moreover, the bounding function is integrable over the distribution of (X, Y) by part 2 of the proposition. Applying the dominated convergence theorem again, we obtain that

$$\beta_j(a_n) = E_Q[f_n(X, Y)] \rightarrow E_Q[\nu'X \cdot m(aY)] = \beta_j(a),$$

as needed. □

D Connection to structural equations models

Previous work has considered a variety of estimators for settings with zero-valued outcomes beginning with a structural equations model rather than the potential outcomes model that we consider. These papers have reached different results, with some concluding that regressions with $\operatorname{arcsinh}(Y)$ have the interpretation of an elasticity, and others showing that they are inconsistent and advocating for other methods (e.g. Poisson regression) instead. In this section, we interpret the results in those papers from the perspective of the potential outcomes model, and show that these diverging conclusions stem from different implicit assumptions about the potential outcomes, as well as a focus on different causal parameters.

Before discussing specific papers, we first note that, broadly speaking, structural equation models can be viewed as constraining the joint distribution of potential outcomes. Observe that, for any pair of potential outcomes $(Y(1), Y(0))$, we can represent them as $(Y(1, U), Y(0, U))$ for some function $Y(d, u)$ and individual-level unobservable (or “structural error”) U . The potential outcomes framework we work with in this paper does not impose any functional form assumptions on $Y(d, u)$. Structural equation models, on the other hand, tend to specify explicit functional forms for $Y(d, u)$. In what follows, we consider the implicit restrictions placed on the potential outcomes as well as the target estimand in work related work that starts with a structural equations model.

D.1 Bellemare and Wichman (2020)

Bellemare and Wichman (2020) consider OLS regressions of the form²²

$$\operatorname{arcsinh}(Y) = \beta_0 + D\beta_1 + U. \tag{4}$$

Note that when D is binary and randomly assigned, $D \perp (Y(1), Y(0))$, then from the perspective of the potential outcomes model, the population coefficient β_1 is the ATE for $\operatorname{arcsinh}(Y)$. Bellemare and Wichman (2020) instead consider the interpretation of β_1 when (4) is treated as structural. From the perspective of the potential outcomes model, this amounts to imposing that the potential

²²They also consider specifications with additional covariates on the right-hand side, although we abstract away from this for expositional simplicity.

outcomes $Y(d) := Y(d, U)$ take the form

$$\operatorname{arcsinh}(Y(d, U)) = \beta_0 + d\beta_1 + U, \quad (5)$$

where the individual-level random variable U takes the same value for all values of d . Under (5), we have that

$$\beta_1 = \operatorname{arcsinh}(Y(1, U)) - \operatorname{arcsinh}(Y(0, U)).$$

Since $\operatorname{arcsinh}(y) \approx \log(2y)$ for y large, it follows that $\beta_1 \approx \log(Y(1, U)/Y(0, U))$ when $Y(1, U)$ and $Y(0, U)$ are large. Thus, [Bellemare and Wichman \(2020\)](#) argue that β_1 approximates the semi-elasticity of the outcome with respect to d when the outcome is large. They likewise provide similar results for the elasticity of $Y(d, U)$ with respect to treatment when treatment is continuous. Their results thus imply that the ATE for $\operatorname{arcsinh}(Y)$ has a sensible interpretation as a (semi-)elasticity when the model for the potential outcomes given in (5) holds.

It is worth emphasizing, however, that (5) will generally be incompatible with the data when both $Y(1)$ and $Y(0)$ have point-mass at zero, and $\beta_1 \neq 0$. Specifically, note that (5) implies that

$$\operatorname{arcsinh}(Y(1, U)) - \operatorname{arcsinh}(Y(0, U)) = \beta_1.$$

If $\beta_1 > 0$, for example, this implies that $\operatorname{arcsinh}(Y(1, U)) > \operatorname{arcsinh}(Y(0, U))$, and hence $Y(1, U) > Y(0, U)$, since the $\operatorname{arcsinh}(y)$ function is strictly increasing for $y \geq 0$. However, if $Y(1, U) = 0$ for some U , this then implies that $Y(0, U) < 0$, which is a contradiction. Thus, the model in (5) is incompatible with $P(Y(1) = 0) > 0$ if $\beta_1 > 0$. By similar logic, the model is also incompatible with $P(Y(0) = 0) > 0$ if $\beta_1 < 0$. In settings where there is point-mass at zero, the model that [Bellemare and Wichman \(2020\)](#) show gives β_1 an interpretation as a semi-elasticity will therefore typically be rejected by the data. It is also worth noting that even if there are no zeros in the data, the model in (5) will generally be sensitive to functional form, in the sense that if (5) holds for Y measured in dollars, it will generally not hold when Y is measured in cents. The validity of the interpretation of β_1 as an elasticity thus depends on having chosen the “correct” scaling of the outcome such that (5) holds.

D.2 Cohn et al. (2022)

[Cohn et al. \(2022\)](#) consider structural equations of the form

$$Y = \exp(D\beta)U. \quad (6)$$

When $E[U \mid D] = 1$, they show that Poisson regression is consistent for β , whereas regressions of $\log(1 + Y)$ or $\log(Y)$ on D may be inconsistent for β .²³ Although [Cohn et al. \(2022\)](#) do not consider

²³Relatedly, in an influential paper, [Silva and Tenreiro \(2006\)](#) consider the structural equations model $Y_i = \exp(X_i'\beta)U_i$ where $E[U_i|X_i] = 1$, and show that Poisson regression consistently estimates β while a regression using \log on the left-hand side does not.

a potential outcomes interpretation of β , we can give β a causal interpretation if we treat (6) as structural, i.e. impose that the potential outcomes take the form

$$Y(d, U) = \exp(d\beta)U, \tag{7}$$

where U is an individual level shock common to all d , and $E[U] = 1$. Under (7), it follows that $\exp(\beta) = E[Y(1)]/E[Y(0)]$, which corresponds with the parameter considered in Section 4.1.²⁴

We note, however, that if one were instead to impose (6) with the assumption that $E[\log(U)|D] = 0$, then the regression of $\log(Y)$ on D would be consistent for β , whereas Poisson regression would generally be inconsistent for β . Indeed, under the potential outcomes model in (7) with the assumption that $E[\log(U)] = 0$, we have that $\beta = E[\log(Y(1)) - \log(Y(0))]$, the ATE in logs.²⁵

This discussion highlights that whether or not an estimator is consistent depends on the specification of the *target parameter*. Our results help to illuminate what parameters can be consistently estimated by enumerating the properties that identified causal parameters can (or cannot) have.

E Connection to two-part models

One approach recommended for settings with weakly-positive outcomes is to estimate a two-part model (Mullahy and Norton, 2022). In this section, we briefly review two-part models, and show that the marginal effects implied by these models do not correspond with ATEs for the intensive margin without further restrictions on the potential outcomes. Thus, while two-part models strike us as a reasonable approach if the goal is to model the conditional expectation function of observed outcomes Y given treatment D (as in Mullahy and Norton (2022)), they will often not be appropriate if instead the goal is to learn about a causal effect along the intensive margin.²⁶

The idea of a two-part model is to separately model the conditional distribution $Y | D$ using (a) a first model for the probability that Y is positive given D , $P(Y > 0 | D)$ (b) a second model for the conditional expectation of Y given that it is positive, $E[Y | D, Y > 0]$. Common specifications include logit or probit for part (a), and a linear regression of Y on D for part b); see, e.g., Belotti, Deb, Manning and Norton (2015). After obtaining estimates of the two-part model, it is common to evaluate the marginal effects of D on both parts, i.e. the implied values of

$$\begin{aligned} \tau_a &= P(Y > 0 | D = 1) - P(Y > 0 | D = 0) \\ \tau_b &= E[Y | Y > 0, D = 1] - E[Y | Y > 0, D = 0]. \end{aligned}$$

We now consider how the parameters of the two-part model relate to causal effects in the

²⁴Bellégo, Benatia and Pape (2022) also consider (6), but consider the more general class of identifying restrictions of the form $E[D \log(U + \delta)] = 0$, where δ is a tuning parameter. The appropriate choice of estimator then depends on δ , with Poisson regression and log regressions the limiting cases as $\delta \rightarrow \infty$ and $\delta \rightarrow 0$, respectively. We note, however, that Bellégo et al. (2022) impose that $E[U] = 1$, and thus the causal interpretation of β in the potential outcomes model in (7) is the same as in Cohn et al. (2022) regardless of the value of δ .

²⁵Note that the assumption that $E[\log(U)] = 0$ implicitly implies that $U > 0$, and thus $Y > 0$.

²⁶We are particularly grateful to John Mullahy for an enlightening discussion of this topic.

potential outcomes model. Suppose, for simplicity, that the two-part model is well-specified, so that it correctly models $P(Y > 0 | D)$ and $E[Y | Y > 0, D]$. Suppose further that D is randomly assigned, $D \perp\!\!\!\perp Y(1), Y(0)$. In this case, we have that

$$\begin{aligned}\tau_a &= P(Y(1) > 0) - P(Y(0) > 0) \\ \tau_b &= E[Y(1) | Y(1) > 0] - E[Y(0) | Y(0) > 0]\end{aligned}$$

From the previous display, we see that the marginal effect on the first margin, τ_a , has a causal interpretation: it is the treatment’s effect on the probability that the outcome is positive.

The interpretation of the marginal effect on the second margin, τ_b , is more complicated however. For simplicity, suppose we are willing to impose the “monotonicity” assumption discussed in [Section 4](#), $P(Y(1) = 0, Y(0) > 0) = 0$, so that anyone with a zero outcome under treatment also has a zero outcome under control. Then, letting $\alpha = P(Y(0) = 0 | Y(1) > 0)$, we can write τ_b as

$$\begin{aligned}\tau_b &= (1 - \alpha)E[Y(1) | Y(1) > 0, Y(0) > 0] + \alpha E[Y(1) | Y(1) > 0, Y(0) = 0] - E[Y(0) | Y(1) > 0, Y(0) > 0] \\ &= \underbrace{E[Y(1) - Y(0) | Y(1) > 0, Y(0) > 0]}_{\text{Intensive margin effect}} + \alpha \underbrace{(E[Y(1) | Y(1) > 0, Y(0) = 0] - E[Y(1) | Y(1) > 0, Y(0) > 0])}_{\text{Selection term}},\end{aligned}$$

where the first equality uses iterated expectations, and the second re-arranges terms.

The previous display shows that τ_b is the sum of two terms. The first is the ATE for individuals who would have a positive outcome regardless of treatment status (similar to $\theta_{\text{Intensive}}$ in [Section 4](#), except using Y instead of $\log(Y)$). The second term is not a causal effect, but rather represents a selection term: it is proportional to the difference in the average value of $Y(1)$ for individuals who would have positive outcomes only under treatment versus individuals who would have positive outcomes regardless of treatment status. In many economic contexts, we may expect this selection effect to be negative. For example, we may suspect that individuals who would only get a job if they receive a particular training have lower ability, and hence lower values of $Y(1)$, than individuals who would have a job regardless of training status. The marginal effect τ_b thus only has an interpretation as an ATE along the intensive margin if either (a) there is no extensive margin effect ($\alpha = 0$) or (b) we are willing to assume that the selection term is zero. [Angrist \(2001\)](#) provided a similar decomposition (without imposing monotonicity), concluding that the two-part model “seems ill suited for causal inference,” at least without further restrictions on the potential outcomes. See, also, [Mullahy \(2001\)](#) for additional discussion.

F Appendix Tables

Paper	Interprets Units as Percent	Quote About Percents / Notes	Original Units
Azoulay et al (2019)	Yes	"In this case, coefficient estimates can be interpreted as elasticities, as an approximation."	Publications (yearly)
Beerli et al (2021)	Yes	"The estimates thus reflect an approximate percentage increase."	Patent applications (yearly)
Berkouwer and Dean (2022)	Yes	"A 0.50 IHS reduction corresponds to a 39 percent reduction relative to the control group."	Weekly expenditure (dollars)
Cabral et al (2022)	Yes	Refers to estimates as "the elasticities reported in panel A"	Costs (dollar) per \$10K risk-adjusted payroll
Carranza et al (2022)	Yes	"Weekly earnings increase by 34% (Table 1, column 3)"	Hours worked (weekly)
Faber & Gauber (2019)	Yes	"A one standard deviation increase in tourism attractiveness increases local manufacturing GDP by about 40 percent."	Municipality GDP (1000s of Pesos)
Hjort and Poulsen (2019)	Yes	"We find that cable arrival increases measured speed in connected locations, relative to unconnected locations, by around 35 percent"	KB per second
Johnson (2020)	Yes	"[T]he regression coefficient estimates the ITT effect of a press release on the percent change in the number of violations. The point estimate (-0.18) is identical to the baseline estimate in percent terms (-0.40/2.29 = 17.5%)."	Violations (monthly)
Mirenda et al (2022)	Yes	"The amount of public funds awarded raises by 3.4 percent."	Contract size (euros)
Norris et al (2021)	Yes	"We measure both the extensive margin (using a binary indicator for the outcome ever occurring) and the intensive margin (taking the inverse hyperbolic sine, IHS, of the number of times the outcome occurred, so the coefficient is interpreted as a percent change)"	Criminal charges
Ager et al (2021)	No interpretation		Wealth (1870 Dollars)
Arora et al (2021)	No interpretation		Publications (yearly)
Bastos et al (2018)	No interpretation		Sales (yearly, euros)
Fetzer et al (2021)	No interpretation		Incidents (quarterly)
Moretti (2021)	No interpretation		Patents (yearly)
Rogall (2021)	No interpretation		Perpetrators
Cao and Chen (2022)	No	They compute $\exp(\beta) - 1$ and multiply by the baseline mean, then interpret this as the effect in levels	Rebellions per million population in 1600

Appendix Table 1: Papers in *AER* estimating effects for $\text{arcsinh}(Y)$ with selected quotes

Note: this table lists papers in the *AER* estimating treatment effects for $\text{arcsinh}(Y)$. The second column classifies papers by whether they interpret the units of the treatment effect as a percent/elasticity, with categories “yes”, “no”, or “no interpretation given.” The third column provides selected quotes and notes about the interpretation of the estimates, and the final column describes the units of the outcome before applying the arcsinh transformation. See [Section 2.3](#) for details.